

# An Introduction to Coordinate-free Quantization and its Application to Constrained Systems<sup>1</sup>

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## Abstract

Canonical quantization entails using Cartesian coordinates, and Cartesian coordinates exist only in flat spaces. This situation can either be questioned or accepted. In this paper we offer a brief and introductory overview of how a flat phase space metric can be incorporated into a covariant, coordinate-free quantization procedure involving a continuous-time (Wiener measure) regularization of traditional phase space path integrals. Additionally we show how such procedures can be extended to incorporate systems with constraints and illustrate that extension for special systems.

## Introduction

In order to quantize a system with constraints it is of course first necessary to have a quantization procedure for systems without constraints. Although the quantization of systems without constraints would seem to be well in hand due to the pioneering work of Heisenberg, Schrödinger, and Feynman, it is a less appreciated fact that all of the standard methods of quantization are consistent only in Cartesian coordinates [1]. As a consequence it follows that the usual quantization procedures depend—or at least seem to depend—on choosing the right set of coordinates before promoting  $c$ -numbers to  $q$ -numbers. This circumstance gives rise to an apparently unwanted coordinate dependence on the very process of quantization. For systems without constraints this is generally not a major problem because an underlying Euclidean space expressed in terms of Cartesian coordinates can generally be assumed. However, for systems with constraints, the configuration space—let alone the frequently more complicated phase space—are generally incompatible with a flat Euclidean structure needed to carry Cartesian coordinates. Hence, before we can properly quantize systems with constraints it will be necessary for us to revisit the quantization of systems without constraints in order to present a coordinate-free procedure for such cases. Only then will we be able to undertake the program represented by the title to this contribution.

All individuals engaged in quantization have a natural inclination to seek a quantization procedure that is as coordinate independent as possible, and in particular, does not

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<sup>1</sup>To appear in the Proceedings of the 2nd Jagna Workshop, Jagna, Bohol, Philippines, January, 1998.

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depend on using Cartesian coordinates inasmuch as the use of such coordinates seems to contradict the ultimate goal of a coordinate-free approach. When faced with the need for quantizing in Cartesian coordinates a number of workers currently seek alternative quantization schemes which avoid completely any reference to Cartesian coordinates. The schemes of geometric quantization and of deformation quantization, among possibly other approaches as well, fit into the category of efforts to eliminate the central role played by Cartesian coordinates, and indeed to construct a fully coordinate independent formulation. It is entirely natural to presume that quantization should be coordinate independent, and so this approach is most reasonable. While these disciplines can be considered and analyzed from a mathematical viewpoint without any conceptual difficulty, it is not a priori evident that just because these methods have the word “quantization” in their name that they have an automatic connection with physics. Indeed, it may be argued that this is not always the case, and this conclusion pertains to the fact that the result does not in general agree with the results of ordinary quantization in the physical sense of the term. Thus such contemporary methods are acceptable as mathematical exercises but should not be taken necessarily as leading to a coordinate-free formulation of quantization as it is needed and used in physics.

Such a circumstance naturally leads to the question: Can we find a coordinate-free form of quantization that does agree with physics, i.e., as a test case to quantize the anharmonic oscillator in accord with the usual quantum mechanical result as obtained, say, from the Schrödinger prescription? The answer in our opinion is yes, and the first part of this article is devoted to a brief review of that subject. The extension of that procedure to systems with constraints is currently in progress, and a preliminary account of part of that work forms the second part of this article.

## Coordinate-free Quantization

In 1948 Feynman proposed the path integral in Lagrangian form, and in 1951 he extended the path integral to a phase space form. It is generally acknowledged that the phase space formulation is more widely applicable than the Lagrangian formulation, and it is the phase space version on which we shall focus. In particular, the expression for the usual propagator is formally given by

$$\mathcal{N} \int \exp\{i \int [p\dot{q} - h(p, q)] dt\} \mathcal{D}p \mathcal{D}q . \quad (1)$$

As it stands, however, this expression is formal and needs to be properly defined. There are several ways to do so. One way involves what may be referred to as a continuous-time regularization in the form [2]

$$\begin{aligned} & K(p'', q'', T; p', q', 0) \\ &= \lim_{\nu \rightarrow \infty} \mathcal{N} \int \exp\{i \int [p\dot{q} - h(p, q)] dt\} \exp[-(1/2\nu) \int (\dot{p}^2 + \dot{q}^2) dt] \mathcal{D}p \mathcal{D}q. \end{aligned} \quad (2)$$

In this expression we have introduced an additional term in the integrand which is formally set to unity in the limit  $\nu \rightarrow \infty$ . However, that extra factor in the integrand serves as

a *regularizing factor*, and this fact can be seen in the alternative—and mathematically precise—expression given by

$$K(p'', q'', T; p', q', 0) = \lim_{\nu \rightarrow \infty} 2\pi e^{\nu T/2} \int \exp\{i \int [pdq - h(p, q)dt]\} d\mu_W^\nu(p, q) , \quad (3)$$

where  $\mu_W^\nu$  denotes a Wiener measure on a flat phase space expressed in Cartesian coordinates and in which  $\nu$  denotes the diffusion constant. In addition, the nature of the regularization forces one to pin (i.e., fix) the values of both  $p$  and  $q$  at the *initial and final times*, namely,  $p'' = p(T), q'' = q(T)$  and  $p' = p(0), q' = q(0)$ . This leads to a nontraditional representation of the propagator, and as we assert below the very regularization itself leads to a *canonical coherent state representation*. Observe, moreover, that (3) is mathematically well-defined and totally ambiguity free. As such, there can be no ambiguity in factor ordering within the quantization procedure at this point, and it is significant that the very regularization chosen has *selected*—or perhaps even better, *preselected*—a particular operator ordering, namely antinormal ordering.

More specifically, on the basis of being a positive-definite function, one may show that

$$K(p'', q'', T; p', q', 0) \equiv \langle p'', q'' | e^{-i\mathcal{H}T} | p', q' \rangle , \quad (4)$$

$$|p, q\rangle \equiv e^{-iqP} e^{ipQ} |0\rangle , \quad (Q + iP) |0\rangle = 0 , \quad \langle 0|0\rangle = 1 , \quad (5)$$

$$[Q, P] = i\mathbb{1} , \quad (6)$$

$$\mathbb{1} = \int |p, q\rangle \langle p, q| dp dq / 2\pi , \quad (7)$$

$$\mathcal{H} = \int h(p, q) |p, q\rangle \langle p, q| dp dq / 2\pi , \quad (8)$$

the last formula being an alternative expression for antinormal ordering. We do not attempt to prove these remarks here; for that the reader may consult the literature [3]. Instead, we limit our discussion to an overview of the general scheme. In that line it is important to note the behavior of these expressions under a *canonical coordinate transformation*. In the classical theory, we often let

$$rds = pdq + dF(s, q) \quad (9)$$

symbolize a canonical change of coordinates where the function  $F$  serves as the “generator” of the coordinate change. In the quantum theory, as described here, the paths  $p(t)$  and  $q(t)$  represent sample paths of a Wiener process, i.e., Brownian motion paths. As such these paths are continuous but nowhere differentiable, and thus they are more singular than the classical path behavior (e.g.,  $C^2$ ) for which (9) normally holds. The integral  $\int pdq$  appearing in (3) is initially undefined due to the distributional nature of the paths involved. There are two standard prescriptions to deal with such stochastic integrals, one due to Itô (I), the other due to Stratonovich (S) [4]. The two prescriptions may be characterized by continuum limits of two distinct discretization procedures. If  $p_l \equiv p(l\epsilon)$  and  $q_l \equiv q(l\epsilon)$  for  $l \in \{0, 1, 2, 3, \dots\}$ , then

$$\int_I pdq = \lim_{\epsilon \rightarrow 0} \sum p_l (q_{l+1} - q_l) , \quad \int_S pdq = \lim_{\epsilon \rightarrow 0} \sum \frac{1}{2} (p_{l+1} + p_l) (q_{l+1} - q_l) . \quad (10)$$

Generally, the results of these two approaches disagree, and it is a feature of the Itô prescription (“nonanticipating”) that the rules of ordinary calculus are generally not obeyed. The Itô prescription has other virtues, but they are not of interest to us here. Instead, we adopt the Stratonovich (“midpoint”) prescription because it possesses the important feature that the ordinary laws of the classical calculus do in fact hold for Brownian motion paths. In particular, therefore, (9) also holds for the paths that enter the regularized form of the phase space path integral, and as a consequence, we find that

$$\int p dq = \int r ds - \int dF(s, q) \equiv \int r ds + \int dG(r, s) = \int r ds + G(r'', s'') - G(r', s') . \quad (11)$$

If we couple this relation with the fact that  $h$  transforms as a *scalar*, i.e., namely, that

$$\bar{h}(r, s) \equiv h(p(r, s), q(r, s)) = h(p, q) , \quad (12)$$

then we learn that under a canonical change of coordinates, the coherent state propagator becomes

$$\begin{aligned} \bar{K}(r'', s'', T; r', s', 0) &\equiv e^{i[G(r'', s'') - G(r', s')]} K(p(r'', s''), q(r'', s''), T; p(r', s'), q(r', s'), 0) \\ &= \lim_{\nu \rightarrow \infty} 2\pi e^{\nu T/2} \int \exp\{i \int [r ds + dG(r, s) - \bar{h}(r, s)] dt\} d\bar{\mu}_W^\nu(r, s) . \end{aligned} \quad (13)$$

Here  $\bar{\mu}_W^\nu(r, s)$  denotes the Wiener measure on a flat two-dimensional phase space no longer expressed, in general, in Cartesian coordinates but rather in curvilinear coordinates. Observe that the form of this expression is exactly that as given in the original coordinates apart from the presence of the total derivative  $dG$ , which leads to nothing more than a phase change for the coherent states. In particular, based on the positive-definite nature of the transformed function we may conclude that

$$\bar{K}(r'', s'', T; r', s', 0) = \langle r'', s'' | e^{-i\mathcal{H}T} | r', s' \rangle , \quad (14)$$

$$|r, s\rangle \equiv e^{-iG(r, s)} e^{-iq(r, s)P} e^{ip(r, s)Q} |0\rangle , \quad (Q + iP)|0\rangle = 0 , \quad \langle 0|0\rangle = 1 , \quad (15)$$

$$\mathbb{1} = \int |r, s\rangle \langle r, s| dr ds / 2\pi , \quad (16)$$

$$\mathcal{H} = \int \bar{h}(r, s) |r, s\rangle \langle r, s| dr ds / 2\pi . \quad (17)$$

Observe carefully that the coherent states have *not* changed under the coordinate transformation, only their *names* have changed. In addition, the operator  $\mathcal{H}$  has *not* changed, only the functional form of the (lower) symbol associated with it has changed. Thus we have achieved a completely covariant formulation of quantum theory!

As an example of such a coordinate change, we may cite the simple case of the harmonic oscillator for which  $h(p, q) = (p^2 + q^2)/2$ . If we introduce new canonical coordinates (action angle variables) according to  $r = (p^2 + q^2)/2$  and  $s = \tan^{-1}(q/p)$ —namely, where  $F(q, s) = -q^2 \cot(s)/2$  and  $G(r, s) = r \cos(s) \sin(s)$ —then it follows that

$$\frac{1}{2}(P^2 + Q^2 + 1) = \int \frac{1}{2}(p^2 + q^2) |p, q\rangle \langle p, q| dp dq / 2\pi = \int r |r, s\rangle \langle r, s| dr ds / 2\pi , \quad (18)$$

which clearly illustrates the fact that although a classical coordinate change has been carried out, all quantum operators, such as  $Q$ ,  $P$ , and particularly the Hamiltonian  $\mathcal{H}$ , have remained completely unchanged!

The foregoing scenario may be readily extended to deal with multiple degrees of freedom, say  $N$ , and—reverting to the original Cartesian coordinates—we find (using the summation convention) that

$$\langle p'', q'' | e^{-i\mathcal{H}T} | p', q' \rangle \equiv \lim_{\nu \rightarrow \infty} (2\pi)^N e^{N\nu T/2} \int \exp\{i \int [p_j dq^j - h(p, q) dt]\} d\mu_W^\nu(p, q) , \quad (19)$$

$$|p, q\rangle \equiv e^{-iq^j P_j} e^{ip_j Q^j} |0\rangle , \quad (Q^j + iP_j)|0\rangle = 0 , \quad \langle 0|0\rangle = 1 , \quad (20)$$

$$[Q^j, P_k] = i\delta_k^j \mathbb{1} , \quad (21)$$

$$\mathbb{1} = \int |p, q\rangle \langle p, q| d\mu_N(p, q) , \quad (22)$$

$$\mathcal{H} = \int h(p, q) |p, q\rangle \langle p, q| d\mu_N(p, q) , \quad (23)$$

$$d\mu_N(p, q) \equiv \Pi_{j=1}^N dp_j dq^j / 2\pi , \quad (24)$$

and where we have used the notation  $p = \{p_j\}_{j=1}^N$  and  $q = \{q^j\}_{j=1}^N$ . We next turn our attention to the inclusion of constraints.

## Constraints and the Projection Method

### Classical preliminaries

From the classical point of view some of the equations of motion that follow from an action principle are just exactly *not* equations of motion in that they do not involve time derivatives but rather conditions that must be satisfied among the canonical variables. Consider a classical phase space action principle of the form

$$I = \int [p_j \dot{q}^j - h(p, q) - \lambda^a \phi_a(p, q)] dt , \quad (25)$$

where  $\lambda^a = \lambda^a(t)$  denote Lagrange multipliers and  $\phi_a(p, q)$  denote constraints, and  $1 \leq a \leq K \leq 2N$ . Stationary variation with respect to the dynamical variables  $p_j$  and  $q^j$  leads to the equations

$$\dot{q}^j = \partial h(p, q) / \partial p_j + \lambda^a \partial \phi_a(p, q) / \partial p_j = \{q^j, h(p, q)\} + \lambda^a \{q^j, \phi_a(p, q)\} , \quad (26)$$

$$\dot{p}_j = -\partial h(p, q) / \partial q^j - \lambda^a \partial \phi_a(p, q) / \partial q^j = \{p_j, h(p, q)\} + \lambda^a \{p_j, \phi_a(p, q)\} , \quad (27)$$

where the last version of each equation is written in terms of Poisson brackets. In turn, stationary variation with respect to the Lagrange multipliers leads to the constraint equations

$$\phi_a(p, q) = 0 \quad (28)$$

the fulfillment of which defines the *constraint hypersurface*. All processes, dynamics included, takes place on the constraint hypersurface. It follows that

$$\dot{\phi}_a(p, q) = \{\phi_a(p, q), h(p, q)\} + \lambda^b \{\phi_a(p, q), \phi_b(p, q)\} = 0 . \quad (29)$$

Assuming that the set  $\{\phi_a\}$  is a complete set of the constraints, two possible scenarios may hold. In the first scenario

$$\{\phi_a(p, q), \phi_b(p, q)\} = c_{ab}{}^c \phi_c(p, q) , \quad (30)$$

$$\{\phi_a(p, q), h(p, q)\} = h_a{}^b \phi_b(p, q) . \quad (31)$$

This situation, termed *first class constraints*, implies that if the constraints are satisfied at one time then they will be satisfied for all time in the future as a consequence of the equations of motion. Observe in this case that the time dependence of the Lagrange multipliers  $\{\lambda^a\}$  is not determined by these equations. To solve the equations for the variables  $p_j(t)$  and  $q^j(t)$  it is necessary to choose the Lagrange multipliers which then constitutes a “gauge choice”. Nothing that is deemed physical can depend on just which gauge choice has been selected, and any observable, say  $O(p, q)$ , must satisfy the relation

$$\{\phi_a(p, q), O(p, q)\} = o_a{}^b \phi_b(p, q) . \quad (32)$$

In the second situation, (30) fails, or (30) and (31) both fail, to hold, and as a consequence, the Lagrange multipliers must assume a special time dependence in order to satisfy (29). In short, in this case, consistency of the equations of motion determines the Lagrange multipliers. This case is termed *second class constraints*. Of course, one may also have a mixed case composed of some first and some second class constraints. In this case some of the Lagrange multipliers are determined while others are not.

The coefficients  $c_{ab}{}^c$ ,  $h_a{}^b$ , and  $o_a{}^b$  above may also depend on the phase space variables. However, for convenience, we shall restrict attention hereafter to those cases where these coefficients are simply constants.

## Quantization à la Dirac

We next take up the topic of quantization of these systems. For the purposes of the present paper we shall confine our attention to the case of first class constraints. (The case of second class constraints has been discussed elsewhere using the methods of the present paper [5, 6].)

According to Dirac [7], quantization of first class systems proceeds along the following line. First quantize the system as if there were no constraints, namely, introduce kinematical operators  $\{P_j\}$  and  $\{Q^j\}$ , which fulfill (21), and a Hamiltonian operator  $\mathcal{H} = \mathcal{H}(P, Q)$  (modulo some choice of ordering). For a general operator  $W(P, Q)$  adopt the dynamical equation

$$i\dot{W}(P, Q) = [W(P, Q), \mathcal{H}] \quad (33)$$

as usual. The constraint operators are assumed to fulfill commutation relations similar to (30) and (31), namely,

$$[\Phi_a(P, Q), \Phi_b(P, Q)] = ic_{ab}{}^c \Phi_c(P, Q) , \quad (34)$$

$$[\Phi_a(P, Q), \mathcal{H}(P, Q)] = ih_a{}^b \Phi_b(P, Q) . \quad (35)$$

Next impose the constraints to select the physical Hilbert subspace in the form

$$\Phi_a(P, Q) |\psi\rangle_{\text{phy}} = 0 . \quad (36)$$

If zero lies in the discrete spectrum of the constraint operators this equation offers no difficulties, and we shall content ourselves with that case. On the other hand, if zero lies in the continuous spectrum, then some subtleties are involved, and one example of where such issues are discussed is [5]. Observe that (34) and (35) demonstrate the consistency of imposing the constraints and the fact that if they are imposed at one time then they will hold for all subsequent time. This imposition of the constraints at the initial time may be called an *initial value equation*, just as in the classical theory.

## The projection method

We note that the commutation relations among the constraint operators is that of a Lie algebra. (Indeed, including the Hamiltonian and noting (35), we observe that the constraints plus the Hamiltonian also form a Lie algebra.) For present purposes we assume that the group generated by this Lie algebra is compact, and we denote the group elements by

$$e^{i\xi^a \Phi_a(P, Q)} . \quad (37)$$

Let  $\delta\xi$  denote the normalized group invariant measure,  $\int \delta\xi = 1$ , and consider the operator

$$\mathbb{E} \equiv \int e^{i\xi^a \Phi_a(P, Q)} \delta\xi . \quad (38)$$

It is a modest exercise to establish that  $\mathbb{E}^2 = \mathbb{E}^\dagger = \mathbb{E}$ , which are just the criteria that make  $\mathbb{E}$  a projection operator. The fact that

$$e^{i\tau^a \Phi_a(P, Q)} \mathbb{E} = \int e^{i\tau^a \Phi_a(P, Q)} e^{i\xi^a \Phi_a(P, Q)} \delta\xi \equiv \int e^{i(\tau+\xi)^a \Phi_a(P, Q)} \delta\xi = \int e^{i\xi^a \Phi_a(P, Q)} \delta\xi = \mathbb{E} , \quad (39)$$

due simply to the invariance of the measure, establishes that  $\mathbb{E}$  is a projection operator onto the subspace where  $\Phi_a = 0$  for all  $a$ , i.e., a projection operator onto the physical Hilbert subspace. We note further, based on (35), that

$$e^{-i\mathcal{H}T} \mathbb{E} = \mathbb{E} e^{-i\mathcal{H}T} \mathbb{E} = \mathbb{E} e^{-i\mathbb{E}\mathcal{H}\mathbb{E}T} \mathbb{E} , \quad (40)$$

Suppose we consider a formal phase space path integral for a system with the classical action functional (25). The formal path integral reads

$$\mathcal{N} \int \exp\{i\int [p_j \dot{q}^j - h(p, q) - \lambda^a \phi_a(p, q)] dt\} \mathcal{D}p \mathcal{D}q = \langle p'', q'' | \mathbf{T} e^{-i\int [\mathcal{H} + \lambda^a \Phi_a] dt} | p', q' \rangle , \quad (41)$$

which, as written, evidently depends on the choice of the Lagrange multipliers. Now let us impose the quantum version of the initial value equation, namely let us force the system

at the initial time to lie in the physical Hilbert subspace. This we may do by considering the expression

$$\int \langle p'', q'' | \mathbf{T} e^{-i \int [\mathcal{H} + \lambda^a \Phi_a] dt} | \bar{p}', \bar{q}' \rangle \langle \bar{p}', \bar{q}' | \mathbb{E} | p', q' \rangle d\mu_N(\bar{p}', \bar{q}') , \quad (42)$$

which has the effect of projecting the propagator onto the physical subspace at time zero. Using the resolution of unity for the coherent states (22), it is straightforward to determine that

$$\begin{aligned} \langle p'', q'' | \mathbf{T} e^{-i \int [\mathcal{H} + \lambda^a \Phi_a] dt} | \bar{p}', \bar{q}' \rangle \langle \bar{p}', \bar{q}' | \mathbb{E} | p', q' \rangle d\mu_N(\bar{p}', \bar{q}') \\ = \langle p'', q'' | \mathbf{T} e^{-i \int [\mathcal{H} + \lambda^a \Phi_a] dt} \mathbb{E} | p', q' \rangle \\ = \langle p'', q'' | e^{-i \mathcal{H} T} e^{i \tau^a \Phi_a} \mathbb{E} | p', q' \rangle \\ = \langle p'', q'' | e^{-i \mathcal{H} T} \mathbb{E} | p', q' \rangle . \end{aligned} \quad (43)$$

Here we have also used the properties of the Lie group relations (34) and (35) to separate the operators in the exponent into two factors, where  $\{\tau^a\}$  denote parameters made of the the functions  $\{\lambda^a(t)\}$  and the constants appearing in (34) and (35). However, whatever the choice of the Lagrange multipliers  $\{\lambda^a\}$ , i.e., whatever the choice of the parameters  $\{\tau^a\}$ , and as indicated in the last line, *the result is completely independent of the Lagrange multipliers*. That is to say, the propagator projected on the subspace spanned by  $\mathbb{E}$  is already, and automatically, gauge invariant. Thus, for first class constraint systems, all that is necessary to achieve a gauge invariant propagator is to project onto the proper subspace at the initial time.

We may introduce the projection operator  $\mathbb{E}$  also by integrating over the Lagrange multipliers  $\{\lambda^a(t)\}$  with one or another suitable measure. Let  $C(\lambda)$  denote a (possibly complex) normalized measure,  $\int dC(\lambda) = 1$ , with the property that it introduces into the integrand at least one copy of the projection operator  $\mathbb{E}$ ; we say at least one since if two (or more) are introduced the result will be the same because  $\mathbb{E}^2 = \mathbb{E}$ , etc. Hence, the desired propagator on the physical subspace may also be written as

$$\langle p'', q'' | e^{-i \mathcal{H} T} \mathbb{E} | p', q' \rangle = \mathcal{N} \int \exp\{i \int [p_j \dot{q}^j - h(p, q) - \lambda^a \phi_a(p, q)] dt\} \mathcal{D}p \mathcal{D}q dC(\lambda) , \quad (44)$$

where, as already noted, the normalized measure  $C$  is designed to introduce one (or more) projection operators  $\mathbb{E}$  [5, 8].

## Commentary

Readers familiar with the proposal of quantization of systems with first class constraints by Faddeev will note a significant difference in the measure for the Lagrange multipliers. In Faddeev's treatment [9] the measure for the Lagrange multipliers is taken to be  $\mathcal{D}\lambda$ , namely a formally flat measure designed to introduce  $\delta$ -functionals of the classical constraints. Such a choice generally leads to a divergence in some of the remaining integrals due to a nonappearance of the variables conjugate to the constraints. Auxiliary conditions



in the form of dynamical gauge fixing are necessary, along with the attendant Faddeev-Popov (F-P) determinant needed to ensure formal canonical coordinate covariance. As is well known, a global choice of dynamical gauge fixing is generally impossible, which then leads to Gribov ambiguities, and their associated difficulties. All these issues arose from using a different measure for the Lagrange multipliers than that which is chosen in the present paper. In our choice the measure for the Lagrange multipliers is *normalized*,  $\int dC(\lambda) = 1$ , namely, an *average over Lagrange multipliers*, for which any divergence is manifestly impossible! It is of course true that the measure for the classical dynamical variables ( $p$  and  $q$ ) is fixed (formally as  $\mathcal{D}p\mathcal{D}q$ ) by the requirements of consistency, but there is no requirement that the Lagrange multipliers must be integrated just as the classical variables here, namely, from  $-\infty$  to  $\infty$  with a flat weighting. Who says we must enforce the *classical* constraints when doing the *quantum* theory? No one of course, and in a general sense that is the only freedom that has been used to avoid dynamical gauge fixing, F-P determinants, Gribov ambiguities, ghosts, ghosts of ghosts, etc., and all the other machinery of the BRST and BFV formalisms [10, 11]. Our results involve only the original phase space variables as augmented by the Lagrange multipliers, do not in any way entail enlargement or even in many cases a reduction in the number of the original variables, lead to results that are entirely gauge invariant and satisfactory, and at the same time avoid altogether the whole galaxy of issues listed above which are instigated by using a flat measure for the Lagrange multipliers.

## Yang-Mills Type Gauge Models

In this section we illustrate the imposition of constraints for a special class of models which we call Yang-Mills type [12]. In the classical theory, in which we assume we have chosen Cartesian coordinates in phase space, the constraints are taken as

$$\phi_a(p, q) = p_a + A_{ab}{}^c q^b p_c, \quad (45)$$

where the parameters  $A_{ab}{}^c$  are antisymmetric in the indices  $b, c$ . This constraint induces a shift in the  $a$ th coordinate and a rotation in the  $b - c$  plane. Such constraints are broad enough to cover the usual Yang-Mills theories (where  $p$  plays the role of the electrical field strength  $E$  and  $q$  the role of the vector potential  $A$ ). It is clear that such constraints commute among themselves to form a Lie algebra. In the quantum theory the constraint operators are taken to be

$$\Phi_a(P, Q) = P_a + A_{ab}{}^c Q^b P_c, \quad (46)$$

and it is an attractive feature of such constraints that

$$e^{i\Omega^a \Phi_a} |p, q\rangle = |p^\Omega, q^\Omega\rangle, \quad (47)$$

namely that the unitary transformation generated by the constraints takes one coherent state into another coherent state. Here

$$p^\Omega \equiv e^{-\Omega^a \text{ad } \phi_a} p, \quad q^\Omega \equiv e^{-\Omega^a \text{ad } \phi_a} q, \quad (48)$$

where  $\text{ad } \phi_a(\cdot) \equiv \{\phi_a, \cdot\}$  and the exponential is defined by its power series expansion.

The inclusion of such constraints into a dynamical system is quite straightforward. For that goal we first note that

$$\int \langle p'', q'' | e^{-i\mathcal{H}T} e^{i\Omega^a \Phi_a} | p', q' \rangle \delta\Omega = \int \langle p'', q'' | e^{-i\mathcal{H}T} | p'^\Omega, q'^\Omega \rangle \delta\Omega = \langle p'', q'' | e^{-i\mathcal{H}T} \mathbb{E} | p', q' \rangle, \quad (49)$$

which asserts that in order to insert the desired projection operator it is only necessary to average the initial coherent state labels over the gauge transformations they experience. In turn this means that

$$\langle p'', q'' | e^{-i\mathcal{H}T} \mathbb{E} | p', q' \rangle = \lim_{\nu \rightarrow \infty} (2\pi)^N e^{N\nu T/2} \int \exp\{i \int [pdq - h(p, q) dt]\} d\mu_W^\nu(p, q) \delta\Omega, \quad (50)$$

where it is understood that the Wiener paths are pinned initially at  $p^\Omega$  and  $q^\Omega$ . We can make this expression appear more familiar by making a change of variables within the well defined path integral. In particular, we let

$$p(t) \rightarrow e^{\int_t^T ds \omega^a(s) \text{ad } \phi_a} p(t), \quad q(t) \rightarrow e^{\int_t^T ds \omega^a(s) \text{ad } \phi_a} q(t), \quad (51)$$

where the functions  $\omega^a(s)$  are arbitrary save for the condition that

$$\int_0^T \omega^a(s) ds = \Omega^a. \quad (52)$$

Since  $(p'^\Omega)^{-\Omega} \equiv p'$  and  $(q'^\Omega)^{-\Omega} \equiv q'$ , this transformation has the effect of removing any influence of the gauge transformation on the initial labels  $p', q'$ , and instead redistributing that influence throughout the time evolution of the path integral. As a proper coordinate transformation within a well defined path integral, we can readily determine the effect of such a change of variables. In particular, appealing to a formal notation for clarity, we find after such a variable change that (50) becomes

$$\begin{aligned} \langle p'', q'' | e^{-i\mathcal{H}T} \mathbb{E} | p', q' \rangle &= \lim_{\nu \rightarrow \infty} \mathcal{N} \int \exp\{i \int [p_j \dot{q}^j - \omega^a \phi_a(p, q) - h(p, q)] dt\} \\ &\times \exp\{-(1/2\nu) \int [(\dot{p} - \omega^a \{\phi_a, p\})^2 + (\dot{q} - \omega^a \{\phi_a, q\})^2] dt\} \mathcal{D}p \mathcal{D}q \delta\Omega. \end{aligned} \quad (53)$$

In this expression a “new” term has appeared in the classical action that looks like a sum of Lagrange multipliers times the constraints, and drift terms have arisen in the Wiener measure regularization. Note also what this formula states: On the right side is a path integral which superficially depends on the functions  $\{\omega^a(s)\}$ ,  $0 \leq s \leq T$ . On the left side, there is no such dependence. In other words, although the path integral *appears* to depend on  $\{\omega^a\}$ , in fact it does not. Therefore we are free to *average* the right-hand side of (53) over the functions  $\{\omega\}$  and still obtain the desired answer. Let the measure  $C(\omega)$  denote such a measure chosen so as to include the initial average over the variables  $\Omega$  as well, and normalized so that  $\int dC(\omega) = 1$ . The only requirement we impose on this measure is that it introduce, as did the original measure over the variables  $\Omega$ , at least

one projection operator  $\mathbb{E}$ . In this case we find the important phase space path integral representation given by

$$\begin{aligned} & \langle p'', q'' | e^{-i\mathcal{H}T} \mathbb{E} | p', q' \rangle \\ &= \lim_{\nu \rightarrow \infty} \mathcal{N} \int \exp\{i \int [p_j \dot{q}^j - \omega^a \phi_a(p, q) - h(p, q)] dt\} \\ & \times \exp\{-(1/2\nu) \int [(\dot{p} - \omega^a \{\phi_a, p\})^2 + (\dot{q} - \omega^a \{\phi_a, q\})^2] dt\} \mathcal{D}p \mathcal{D}q dC(\omega) . \end{aligned} \quad (54)$$

Here we can really see the variability of the Lagrange multipliers in the path integration and how the proper choice of measure for them can lead to the desired gauge invariant result without any additional complications.

The only topic left to discuss is what should we take for the measure  $C$ . In fact there are many answers to that question, but, for brevity, we shall only indicate one of them. We suppose that our compact gauge group is semisimple and therefore admits a group-induced, positive definite metric  $g_{ab}(\omega)$ . Given that metric on the group manifold we introduce a Wiener measure formally given by

$$dC(\omega) = \mathcal{M} \exp[-\frac{1}{2} \int g_{ab} \dot{\omega}^a \dot{\omega}^b dt] \Pi_t \delta\omega(t) , \quad (55)$$

where this measure is *not* pinned either at  $t = 0$  or at  $t = T$ . A normalized measure without pinning is made possible because the space of variables  $\{\omega^a\}$  at any one time is compact. The formal constant  $\mathcal{M}$  is chosen to ensure  $\int dC(\omega) = 1$ .

In summary, we have illustrated how the use of coherent states and a natural flat phase space metric can be used to develop a coordinate-free quantization procedure for systems without constraints as well as for systems with constraints. We hope this introductory paper may encourage the reader to delve further into this fascinating subject.

## Dedication

It is a pleasure to dedicate this paper to the 65th birthday of Hiroshi Ezawa, which was the main event celebrated at the 2nd Jagna Workshop. Over a number of years, one of the authors (J.R.K.) has enjoyed numerous interactions with the honoree including, but not limited to, two years of close collaboration at Bell Laboratories, and several visits to Tokyo to share in the scientific and social life of Japan. Such great personal interactions and experiences are truly what makes life worthwhile!

## Acknowledgements

It is a great pleasure to thank the organizers Chris and Victoria Bernido, and their extended families, for hosting such a pleasant meeting, and which additionally offered the participants a delightful glimpse of Philippine country life. It was an additional pleasure to meet old friends and to make new ones among the local participants. We hope the series of Jagna workshops will continue for many years to come!

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